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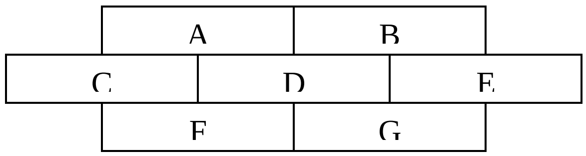
Norfolk, 2001

Homework Set 1 — [the first] Problem 8

8.a. Yes, there exists at least one map of rectangles whose coloring requires three colors.

Consider the following map of seven rectangles, each of which has been labeled for discussion:

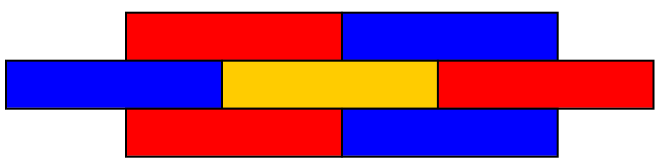


Since each rectangle on the map must eventually receive a color, we are free to select any specific rectangle, *x*, and discuss its color, *n*, without loss of generality — for instance, we observe rectangle D and note that it must be assigned a color which we will call, simply, *color #1*.

Noting that rectangle A shares a border with rectangle D (and remembering that no two rectangles that share a border may be colored alike) we recognize that rectangle A MUST be filled with a color other than *color #1*; we call this second distinct color, *color #2*.

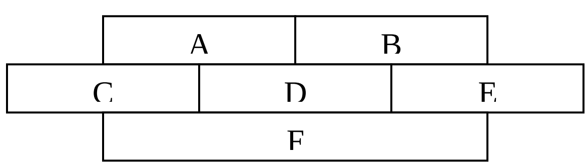
Finally, we note that rectangle B shares a border with both A and D and therefore CANNOT contain either *color #1* or *color #2*. Rectangle B REQUIRES a third color, *color #3*.

At this point, we have shown that AT LEAST three colors are REQUIRED by the map — but are three colors sufficient? We show that three colors are, in fact, sufficient with the graphic below:



8.b. Yes, there exists at least one map of rectangles whose coloring requires four colors.

Consider the following map of six rectangles, each of which has been labeled for discussion:



As in 8.a, above, we begin with rectangle D, with no loss of generality, and call its assigned color, *color #1*. Additionally, we note that rectangles A and B share the same exact relationship as in 8.a. and for the same reasons are REQUIRED to assign them distinct colors, *color #2* and *color #3*, respectively.

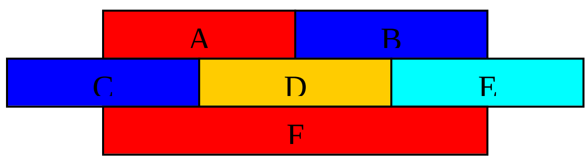
At this point, one of two cases must be true:

1. We can complete the coloring using only the three colors already identified.
2. We will need at least one additional color (at least 4, total) to complete the map.

When we consider rectangle C, we notice that it is already bordered by two distinct colors (*color #1* and *color#2*) and an unknown color in rectangle F. How should we proceed?

If case 1 is true, we must use *color#3* in rectangle C. Then, since C (*color #3*) and D (*color #1*) border F, F must use *color #2*. Since rectangle E is bordered by B, D, and F, it cannot use *color #3*, *color#1*, or *color#2*, respectively; hence, E cannot legally be colored using any of the existing colors and case 1 is proven to be false.

Since case 1 is proven false, case 2 must be true: we need at least four colors to complete the map — but are four colors sufficient? We show that four colors are, in fact, sufficient with the graphic below:



8.c. No, we cannot find a map of rectangles that requires five colors. The Map Coloring Theorem (MCT) states that the maximum number of colors required to color any map is at most four.

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Homework Set 1 — [the second] Problem 8

8. We can transform the given conflict matrix into a graph coloring problem using the following schema:

Problem	Graph representation
Objects which need to be grouped in some way so that there are no conflicts within any group	Vertices — representing the individual students
Pairs of objects which are in conflict	Edges — representing conflicts between pairs of students
The groups	Colors — representing the minivans

In our new schema, the stated question becomes: Are two colors sufficient to color the graph on the following page?

Assumption: The graph can be colored with only two colors.

Since the graph can be colored, vertex D receives a color, call it *color 1*. Since vertex I shares an edge with D, I must receive a different color, *color 2*. Since vertex J shares an edge with I, J must be colored with *color 1*. Since vertex K shares an edge with J, K must be colored with *color 2*. Note that vertex C shares an edge with vertex D (*color 1*) and with vertex K (*color 2*) — it is impossible to color vertex C with the existing colors, so our initial assumption must be false: the graph CANNOT be colored with only two colors. Therefore, two minivans are not sufficient for these twelve students.

Alternately, consider the students and two empty minivans (1) and (2).

- Student D must go on a minivan, so he boards the first empty one (1).
- Student I conflicts with D, must board the other minivan (2).
- Student J conflicts with I, so must go on the first minivan (1).
- Student K conflicts with J, so must go on the second minivan (2).
- Student C conflicts with K (minivan 1) and D (minivan 2), so cannot go onto either minivan without creating a conflict.

Since C cannot board a minivan without creating a conflict, two minivans are not sufficient.

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Homework Set 2 — Problem 6

Our student has made two statements:

1. The vertices of the graph she found each had even degree.
2. The graph she found did not make an Euler circuit.

As demonstrated with the "paper plate" experiment, any time an edge between two vertices is added or removed from a graph, one of three possible outcomes occur:

The total number of vertices with odd degree remains constant.
The total number of vertices with odd degree increases by exactly 2.
The total number of vertices with odd degree decreases by exactly 2.

Since the total number of vertices with odd degree must always change by +2, -2, or 0, we note that it is impossible to add edges to a graph in such a way as to produce an odd number of vertices with odd degree.

Keeping this in mind, we temporarily remove the "connecting edge" that bridges the left and right parts of the paper. (This "connecting edge" is the edge the student claims will prevent us from completing our circuit.) We notice that we now have two graphs, one on the left-hand side of the paper and one on the right-hand side. Examining either graph reveals one vertex with odd degree in each. Since no graph can have an odd number of vertices with odd degree, we are forced to conclude that for each of these new graphs, somewhere beyond the edge of the paper, at least one additional vertex with odd degree must exist. This contradicts the student's claim that all vertices drawn on the [uneaten] paper had even degree.

Since the student's first claim is contradicted, she is mistaken: the graph presented does not "contain only vertices with even degree yet fail to make an Euler circuit."

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Homework Set 2 — Problem 9

9. Sam wishes to show connections between seven distinct objects; since graphs can be used to show connections between objects, we should be able to model his sculpture with a graph. We attempt such a graph, using seven vertices to represent the individual pillars and edges to show the golden threads between them.

Assumption: each of seven pillars can be connected to exactly three others.

Sam has specified that each pillar must be connected to exactly three others, so our graph must show each vertex with a degree of three, thus representing each pillar with exactly three strings emerging.

But since each of seven vertices is required to have a degree of three, we find ourselves with an odd number of vertices with odd degree — seven. As seen in the “paper plate” experiment, no graph can ever contain an odd number of vertices with odd degree, so we cannot model the structure as a graph. Since we cannot model the structure as a graph, then it is impossible that seven pillars can have the specified connections and, therefore, our beginning assumption must be false: each of seven pillars cannot be connected to exactly three others.

— Alternately

We find the number of edges needed for our graph using the “counting legs” equation:

(7 vertices x 3 legs/vertex) ÷ 2 = 10.5 edges

There is no way to draw one-half of an edge connecting two vertices; only whole edges connect two vertices.

Here, again, we find that it is impossible to draw a graph representing the desired connections. Since a graph cannot be drawn, this structure cannot be built.

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Homework Set 3 — Problem 3

3. Prior to answering this problem, we stipulate the following rules to preserve the spirit of the question (i.e., as a collection of introductions):

- No person may shake his or her own hand
- No person may shake hands more than once with the same person

3.a.
Assumption: Four people can each shake a different number of hands.

Given our assumption, the following general assignments must be true:

1. One person must shake no hands.
2. One person must shake one hand.
3. One person must shake two hands.
4. One person must shake three hands.

Note: no person can shake four hands, since four other people are not available and he or she cannot shake his or her own hand to create a fourth.

The person who must shake exactly three hands is required to shake hands with every other person (1-3, above) and, therefore, is required to shake hands with the person who must shake NO hands. Plainly, one person cannot shake everyone else's hand in a room where one of those people will not shake anyone's hand, so statements 1 and 3 cannot both be true and our assumption has to be false.

Each of four people cannot shake a different number of hands than everyone else.

3.b.
Assumption: Five people can each shake a different number of hands.

Given our assumption, the following general assignments must be true:

One person must shake no hands.
One person must shake one hand.
One person must shake two hands.
One person must shake three hands.
One person must shake four hands.

Here, again, one person must shake all others while a different person must shake none. This is logically impossible, so our assumption must be false.

Each of five people cannot shake a different number of hands than everyone else.

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Homework Set 3 — Problem 9

9. When we consider the number of possible ways nickels and/or dimes can be grouped to generate 75 cents, we find the following permissible groupings:

Group	Nickels Used	Dimes Used	Dollar Value	# Coins
A	15	0	75 cents	15
B	13	1	75 cents	14
C	11	2	75 cents	13
D	9	3	75 cents	12
E	7	4	75 cents	11
F	5	5	75 cents	10
G	3	6	75 cents	9
H	1	7	75 cents	8

Consider each group as a string of nickels (N) and dimes (D):

A N N N N N N N N N N N N N N N
B N N N N N N N N N N N N N D
C N N N N N N N N N N N D D
D N N N N N N N N N D D D
E N N N N N N N D D D D
F N N N N N D D D D D
G N N N D D D D D D
H N D D D D D D D

Within each group, the placement of nickels and dimes can be re-arranged to create unique strings that are anagrams of the original string. The number of distinct anagrams in each group is given as *n* C *r*, where *n* represents the number of places in the string, *r* represents the number of times either of our two variables is repeated and *n* C *r* is calculated as *n*! /((*n*-*r*)! *r*!)

A 15 C 0 = 1
B 14 C 1 = 14
C 13 C 2 = 78
D 12 C 3 = 220
E 11 C 4 = 330
F 10 C 5 = 252
G 9 C 6 = 84
H 8 C 7 = 8

The total number of ways to insert dimes and nickels into a vending machine to make 75 cents is the sum of the number of ways to do so in groups A through H.

Total = 1+14+78+220+330+252+84+8= 987

[Alternately, recognize the problem as counting the number of ways to tile ones and twos onto a 15-digit string and, so, take the 15th Fibonacci, which equals 987.]

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Homework Set 4 — Problem 7

7. We break the selection process into two tasks:

A. Select three flavors from the 14 offered.
B. Select two toppings from the 5 offered.

Once we know the number of ways to accomplish task A and the number of ways to accomplish task B, we use the Multiplication Rule for Counting to arrive at the final number of ways to assemble a sundae.

Task A is accomplished by choosing 3 flavors from the 14 offered. Therefore, there are "14 Choose 3" [or 364] ways to accomplish Task A.

Task B is accomplished by choosing 2 toppings from the 5 offered. Therefore, there are "5 Choose 2" [or 10] ways to accomplish Task B.

By the Multiplication Rule for Counting, the total number of ways to assemble a sundae is 364 * 10 = 3640.

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Homework Set 4 — Problem 12

The stated problem contains three specific rules:

A. Use no digit more than once.
B. No odd digit may follow an even digit.
C. The “4” and the “5” must be next to each other.

To construct a ten-digit number without repeating any digits (Rule A) we must use each digit, 0-9, exactly once.

To ensure no odd digit follows an even digit (Rule B) we must place all odd digits in the leftmost five positions and the even digits in the rightmost five positions.

To ensure the “4” and the “5” are next to each other (Rule C), the fifth digit must be the “5” and the sixth digit must be the “4” — remember, all even digits must fall to the right of all odd digits.

A	B	C	D	E	F	G	H	I	J

Treat the 10-digit string as a 10-step task, filling in allowed digits, then use the Multiplication Rule for Counting to find the total number of acceptable strings.

Task	Number of ways to perform
E	Must be the “5”, so only 1 possible way, leaving 4 odd digits
D	Must be one of the remaining odd digits, so 4 possible ways, leaving 3 odd digits
C	Must be one of the remaining odd digits, so 3 possible ways, leaving 2 odd digits
B	Must be one of the remaining odd digits, so 2 possible ways, leaving 1 odd digit
A	Must be the remaining odd digit, so 1 possible way
F	Must be the “4”, so only 1 possible way, leaving 4 even digits
G	Must be a remaining even digit, so 4 possible ways, leaving 3 even digits
H	Must be a remaining even digit, so 3 possible ways, leaving 2 even digits
I	Must be a remaining even digit, so 2 possible ways, leaving 1 even digit
J	Must be the remaining even digit, so 1 possible way

A	B	C	D	E	F	G	H	I	J

By the Multiplication Rule for Counting, there are 1*2*3*4*1*1*4*3*2*1 = 576 10-digit numbers that satisfy the required conditions.

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Homework Set 5 — Problem 6

The 10 employees must fill exactly 10 positions. Several possible examples are shown, below, where a letter identifies each of the employees, A-J:

Employee	A	B	C	D	E	F	G	H	I	J
Pos. Ex. 1	Stock	Sales	Sales	Sales	Sales	Stock	Clean	Clean	Stock	Cashier
Pos. Ex. 2	Clean	Clean	Stock	Stock	Stock	Sales	Sales	Sales	Sales	Cashier
Pos. Ex. 3	Cashier	Clean	Clean	Stock	Stock	Stock	Sales	Sales	Sales	Sales

The anagram counting method tells us that the answer to our question can be found by taking the factorial of the number of positions, *n*, and dividing by the factorials of the number of times each position must occur.

In this case, *n* is 10, since our selection "string" is ten employees long.

The number of times each position must occur is given below:

Cashier	1 time
Cleaner	2 times
Stocker	3 times
Salesman	4 times

Therefore, the anagram counting method tells us that there are

10! / (1! * 2! * 3! * 4!) = 12,600 unique ways to assign the ten employees in the shop.

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Homework Set 5 — Problem 12

The number of five-card hands meeting our requirement may be calculated using the Multiplication Rule for Counting, where the selection of each of the five cards is a necessary task toward building the final hand. The number of hands generated is the number of elements in the event set of our problem.

The total number of ALL five-card hands possible can also be calculated using the Multiplication Rule for Counting. The hands form the outcome set of our problem.

The probability of being dealt a five card hand containing no pairs is calculated by dividing the number of events in the event set (# possible hands containing no pairs) by the number of outcomes in the outcome set (# all possible hands).

Use the string _____ _____ _____ _____ _____ to represent our five-card hand.
A B C D E

Hands in the event set:

- Our first card, A, can be any card, without restriction, so there are 52 possible selections.
- Our second card, B, cannot match the first card, so we discard the three remaining cards with that index, leaving us 48 possible cards from which to select card B.
- As with the second card, our third card, C, cannot match the index of any previous cards, so we discard the three cards that share B's index, giving us 44 cards from which to select C.
- By similar reasoning, card D must be chosen from 40 cards and card E from 36 cards.

The number of cards in the event set is 52 x 48 x 44 x 40 x 36 = 63,360

Hands in the outcome set:
Our first card, A, can be any card, without restriction, so there are 52 possible selections.
Our second card, B, can be any of the (now) remaining 51 cards.
Our third card, C, can be any one of the (now) remaining 50 cards.
Our fourth card, D, can be any one of the (now) remaining 49 cards.
Our fifth card, E, can be any one of the (now) remaining 48 cards.

The number of cards in the outcome set is 52 x 51 x 50 x 49 x 48 = 124,950

Probability of receiving a hand with no pairs:

$$\frac{\text{Number of cards in the event set}}{\text{Number of cards in the outcome set}} = \frac{63,360}{124,950} = 0.507 \text{ or, roughly, } 51\%.$$

Alternately, use the Multiplication Rule for Probability:

52/52 x 48/51 x 44/50 x 40/49 x 36/48 = 0.507